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ABSTRACT PETER-WEYL THEORY FOR SEMICOMPLETE ORTHONORMAL SETS

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Abstract

The central concept in the harmonic analysis of a compact group is the completeness of Peter-Weyl orthonormal basis as constructed from the matrix coefficients of a maximal set of irreducible unitary representations of the group, leading ultimately to the direct sum decomposition of its L^2- space. A Peter-Weyl theory for a semicomplete orthonormal set is also pos- sible and is here developed in this paper for compact groups. Existence of semicomplete orthonormal sets on a compact group is proved by an explicit construction of the standard Riemann- Lebesgue semicomplete orthonormal set on the Torus, T. This approach gives an insight into the role played by the L^2- space of a compact group, which is discovered to be just an example (indeed the largest example for every semicomplete orthonormal set) of what is called a prime-Parseval subspace, which we proved to be dense in the usual L^2- space, serves as the natural domain of the Fourier transform and breaks up into a direct-sum decomposition. This paper essentially gives the harmonic analysis of the prime-Parseval subspace of a compact group corresponding to any semicomplete orthonormal set, with an introduction to what is expected for all connected semisimple Lie groups through the notion of a $K-$semicomplete orthonormal set.

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§1. Introduction.

Harmonic analysis on a compact group is mainly a direct consequence of the famous *Peter-Weyl theory* which gives a consistent method, via the computation of the matrix coefficients of its *irreducible unitary representations*, of deriving a *complete orthonormal set* which is immediately responsible for the direct-sum decomposition of its L^2- space and *regular representation*. Even though such a complete orthonormal set is non-existence for non-compact topological groups and hence the harmonic analysis on *non-compact topological groups*, as we know for *connected nilpotent* and *semisimple Lie groups*, has had to be developed through other means notably via the differential equations satisfied by the (*spherical*) functions derived as matrix coefficients of irreducible unitary representations constructed from *parabolic* and *cohomological inductions* and the completeness afforded by the *Plancherel theorem* (which in the final analysis still depends on the availability and properties of the *discrete series* (known to be the irreducible unitary representations corresponding to some complete orthonormal set) of some distinguished compact subgroups), it still found to be appropriate (and to have a sense of finality) to have some forms of *Peter-Weyl* results on such *non-compact topological groups*.

It is however possible to get at the decomposition of the regular representation of a compact group G (for a start) via the indirect use of the notion of a *semicomplete orthonormal set* on such a group, leading to the consideration of a distinguished subspace of $L^2(G)$ which is established to be *topologically dense*. The study in this paper opens up this field of research by a detailed look at the *compact case*. The paper is arranged as follows.

§2. contains a quick review of the well-known notion of a complete orthonormal set on a compact group, giving the detailed of the aforementioned consistent way of constructing such a set through *Peter-Weyl theorem* which then leads to the direct-sum decomposition of its L^2–space. The concept of a *semicomplete orthonormal set* on a compact group G is introduced in §3. with constructible examples (prominent among which is the *Riemann-Lebesgue* orthonormal set)on the Torus, $\mathbf{T}$, where we derived and used the properties of the *Fourier* and *prime-Parseval subspaces* of $L^2(G)$. Chief among these properties is the topological denseness of every *prime-Parseval subspace* in $L^2(G)$. This takes us to the *Fourier transform* of the *prime-Parseval subspace* and its

direct-sum decomposition into *invariant subspaces*. The last section gives an introductory extension of the results of §3. on compact groups to connected semisimple Lie groups with finite center.

§2. Fourier and Parseval subspaces for complete orthonormal set.

A mutually orthonormal family $\{\chi_\alpha\}_{\alpha\in A}$ in a Hilbert space, $(H, \langle\cdot,\cdot\rangle)$ is said to be *complete* (in H) if $x \in H$ is such that $\langle x, \chi_\alpha\rangle = 0$ (for every $\alpha \in A$) implies $x = 0$. This means that a family $\{\chi_\alpha\}_{\alpha\in A}$ of mutually orthonormal members of H is complete whenever it can be shown that the zero element of H is the *only* non-member of the family that is mutually orthonormal to all members of the said family. Two other equivalent methods of confirming the completeness of the family $\{\chi_\alpha\}_{\alpha\in A}$ are as follows.

2.1 Lemma. ([5.], p. 3) *Let $\{\chi_\alpha\}_{\alpha\in A}$ denote a mutually orthonormal family in a Hilbert space $(H, \langle\cdot,\cdot\rangle)$. The following are equivalent:*
(a) Every $x \in H$ can be expressed as $x = \sum_{\alpha\in A}\langle x, \chi_\alpha\rangle\chi_\alpha$.
(b) Every $x \in H$ satisfies $\| x \|^2 = \sum_{\alpha\in A} | \langle x, \chi_\alpha\rangle |^2$.
(c) $\{\chi_\alpha\}_{\alpha\in A}$ is complete in H. □

The informed reader would observe that (a) of (2.1) is a *Fourier series* expansion of x while (b) of (2.1) is its *Parseval equality*, both with respect to $\{\chi_\alpha\}_{\alpha\in A}$. The import of this equivalence (in the light of (a) of (2.1) (respectively, (b) of (2.1))) is that every $x \in H$ has a Fourier series expansion in terms of any known complete orthonormal family in H. We could then say that the subset $H(\chi_\alpha)$ of H given as

$$\{x \in H : x = \sum_{\alpha\in A}\langle x, \chi_\alpha\rangle\chi_\alpha, \text{ for some orthonormal family } \{\chi_\alpha\}_{\alpha\in A} \text{ in } H\}$$

(equivalently, the subset $H_{\mathfrak{P}}(\chi_\alpha)$ of H given also as

$$\{x \in H : \| x \|^2 = \sum_{\alpha\in A} | \langle x, \chi_\alpha\rangle |^2, \text{ for some orthonormal family } \{\chi_\alpha\}_{\alpha\in A} \text{ in } H\})$$

is exactly H if, and only if, $\{\chi_\alpha\}_{\alpha\in A}$ is complete. Indeed another version of the equivalence of Lemma 2.1, whose formulation serves as our point of departure, is given as follows.

2.2 Lemma. *Let $\{\chi_\alpha\}_{\alpha\in A}$ denote a mutually orthonormal family in a Hilbert space $(H, \langle\cdot,\cdot\rangle)$. The following are equivalent:*
(a) $H(\chi_\alpha) = H$
(b) $H_{\mathfrak{P}}(\chi_\alpha) = H$
(c) $\{\chi_\alpha\}_{\alpha\in A}$ is complete in H. □

2.3 Remarks. It may be safely conjectured that the *Fourier subspace* $H(\chi_\alpha)$ as well as the *Parseval subspace* $H_{\mathfrak{P}}(\chi_\alpha)$ (of a Hilbert space H) with respect to a complete mutually orthonormal family will always be equal to H. It will be a delight to study the disparity between the *Fourier subspace* $H(\chi_\alpha)$ as well as the *Parseval subspace* $H_{\mathfrak{P}}(\chi_\alpha)$ (of H with respect to the mutually orthonormal family $\{\chi_\alpha\}_{\alpha\in A}$) and their inclusions in H, when the family $\{\chi_\alpha\}_{\alpha\in A}$ is not complete.

For example, if the family $\{\chi_\alpha\}_{\alpha\in A}$ of mutually orthonormal members in H is such that $\langle x, \chi_\alpha\rangle = 0$ *(for every $\alpha \in A$) does not necessarily imply whether $x = 0$ or $x \neq 0$*, it possible to then have that

$$0 \leq \| x \|^2 = \sum_{\alpha\in A} | \langle x, \chi_\alpha\rangle |^2 = 0,$$

showing in this case (for the family $\{\chi_\alpha\}_{\alpha\in A}$ in which $\langle x, \chi_\alpha\rangle = 0$ (for every $\alpha \in A$) does not necessarily imply whether $x = 0$ or $x \neq 0$) that we now have $H_{\mathfrak{P}}(\chi_\alpha) = \{0\}$ $(= H(\chi_\alpha) \neq H$, showing that both subspaces are too small and far from being equal to H). This shows at a glance the importance of completeness of the family $\{\chi_\alpha\}_{\alpha\in A}$ in the consideration of the *Parseval equality*, for the non-triviality of these two subspaces $H(\chi_\alpha)$ and $H_{\mathfrak{P}}(\chi_\alpha)$ and for the sustenance of the relationship of equality (of Lemma 2.2) between $H(\chi_\alpha)$ and $H_{\mathfrak{P}}(\chi_\alpha)$. □

However, and as it shall be shown in the next section, these two subspaces, $H(\chi_\alpha)$ and $H_{\mathfrak{P}}(\chi_\alpha)$ may be considered for an *appropriately chosen* not-necessarily complete orthonormal family $\{\chi_\alpha\}_{\alpha\in A}$ and with which they would still be found not to be too small in sizes (in comparison with H). This choice of a not-necessarily complete orthonormal family $\{\chi_\alpha\}_{\alpha\in A}$ would equally help and be appropriate in order that both $H(\chi_\alpha)$ and $H_{\mathfrak{P}}(\chi_\alpha)$ be *lifted* to all of H. All this in a moment.

A well-known method of computing complete orthonormal family of functions is via the matrix coefficients of irreducible unitary representations of a compact groups G which is then used to decompose $L^2(G)$ into invariant subspaces, leading to the decomposition of the right regular representation on G (which sadly, does not generalize to *non-compact topological groups*). Here is the technique.

Denote the *dual* of a compact group G by $\widehat{G}$, consisting of all its equivalence classes of irreducible unitary representations. For $\lambda \in \widehat{G}$ denote by u_{ij}^{λ} the corresponding matrix coefficient representative of the class λ whose

degree is also denoted by $d(\lambda)$. Then the set

$$\{\sqrt{d(\lambda)}u_{ij}^{\lambda} : \lambda \in \widehat{G},\ 1 \le i,j \le d(\lambda)\}$$

consists of a maximal set of complete orthonormal family of functions in $L^2(G)$ and (hence) every $f \in L^2(G)$ can be expanded as

$$f = \sum_{\lambda \in \widehat{G}} d(\lambda) \sum_{i,j}^{d(\lambda)} \langle f, u_{ij}^{\lambda} \rangle u_{ij}^{\lambda}$$

(with convergence in the norm of $L^2(G)$) whose *Fourier transform*

$$\widehat{f} : \widehat{G} \to M_{d(\lambda)}(\mathbb{C}) : \lambda \mapsto \widehat{f}(\lambda) = (\widehat{f}(\lambda)_{ij})_{i,j=1}^{d(\lambda)}$$

is given as $\widehat{f}(\lambda)_{ij} := \langle f, u_{ij}^{\lambda} \rangle$ (where $M_{d(\lambda)}(\mathbb{C})$ denotes the algebra of matrices with entries in $\mathbb{C}$ and degree $d(\lambda)$). It then follows that for any compact group G, the *Fourier subspace* $L^2(G)(\sqrt{d(\lambda)}u_{ij}^{\lambda})$ of $L^2(G)$ is given as

$$L^2(G)(\sqrt{d(\lambda)}u_{ij}^{\lambda}) := \{f \in L^2(G) :\ f = \sum_{\lambda \in \widehat{G}} d(\lambda) \sum_{i,j}^{d(\lambda)} \langle f, u_{ij}^{\lambda} \rangle u_{ij}^{\lambda}\} = L^2(G)$$

$(= L^2(G)_{\mathfrak{P}}(\sqrt{d(\lambda)}u_{ij}^{\lambda})$, the *Parseval subspace* of $L^2(G)$), with respect to the family $\{\sqrt{d(\lambda)}u_{ij}^{\lambda} :\ \lambda \in \widehat{G},\ 1 \le i,j \le d(\lambda)\}$. We then have the abstract direct-sum decomposition of $L^2(G)$ given as

$$L^2(G) = \bigoplus_{\lambda \in \widehat{G}} \bigoplus_{i=1}^{d(\lambda)} H_i^{\lambda},$$

where $H_i^{\lambda} := \sum_{j=1}^{d(\lambda)} \mathbb{C}u_{ij}^{\lambda}$. This is the content of *Peter-Weyl Theorem*, [5.], and we shall refer to the set

$$\{\sqrt{d(\lambda)}u_{ij}^{\lambda} :\ \lambda \in \widehat{G},\ 1 \le i,j \le d(\lambda)\}$$

as the *standard Peter-Weyl orthonormal set* on G.

The inability of being able to get an orthonormal family in $L^2(G)$ for a non-compact topological group G in the above tradition of Peter-Weyl is the first stumbling block to harmonic analysis on such groups, which has been

considerably understood and completely developed via a rigorous treatment of the rich structure of differential equations satisfied by matrix-coefficients of members of each of the classes in $\widehat{G}$, [2]. This paper presents a constructive method of getting a not-necessarily complete orthonormal set which is *close enough* to being a complete orthonormal family in an arbitrary Hilbert space $(H, \langle\cdot,\cdot\rangle)$ and/or in $L^2(G)$, for a compact group (and introduced the same technique for a semisimple Lie group) offering a more general Fourier series expansion of each member of an appropriate subspace of H and/or $L^2(G)$.

Starting with a compact group (before extending the notion to all connected semisimple Lie groups, with finite center, via its *Iwasawa decomposition*) we would however not approach harmonic analysis on the groups via the completeness (and consequent denseness) of the *standard Peter-Weyl orthonormal set,* but via a denseness in the L^2–space which would be found to be possible from an *almost complete* orthonormal set.

§3. Semicomplete orthonormal set in a compact group.

The existence of different special functions and polynomials of mathematical physics, which have been established to be orthonormal in various semisimple Lie groups (compact and non-compact types), is well-known. However the absence of completeness of these orthornormal families (under the structure of their individual corresponding groups) is the first stumbling block to a direct *Peter-Weyl harmonic analysis* of them. In this section we shall define and study the concept of a *semicomplete* orthonormal family in a compact group in order to extend this concept to the harmonic analysis of *all* semisimple Lie groups in the next section.

3.1 Definition. (*Semicomplete orthonormal family*) *Let G denote a compact group and let the members of the non-empty set A be ordered such that* $A = \{\alpha_i^j\}_{i,j}$. *An orthonormal family* $\{\chi_{\alpha_i^j}\}_{\alpha_i^j \in A}$ *in* $L^2(G)$ *is said to be semicomplete if given* $\epsilon > 0$ *there exist some non-zero scalars*

$$\gamma_1, \cdots, \gamma_k, \cdots, \beta_{11}, \cdots, \beta_{ij}, \cdots \in \mathbf{C}$$

and $n \in \mathbf{N}$ *such that*

$$\| \sum_{\lambda \in \widehat{G}} d(\lambda) \sum_{i,j=1}^{d(\lambda)} (f, u_{ij}^{\lambda}) u_{ij}^{\lambda} - \sum_{j=1}^{n} \gamma_j \sum_{i=1}^{n} \beta_{ij} \sum_{\alpha_i^j \in A} (f, \chi_{\alpha_i^j}) \chi_{\alpha_i^j} \|_2 < \epsilon$$

for every $f \in L^2(G)$. □

The quantity

$$\sum_{\lambda\in\widehat{G}} d(\lambda) \sum_{i,j=1}^{d(\lambda)} (f, u_{ij}^{\lambda}) u_{ij}^{\lambda}$$

in Definition 3.1 above may be replaced with f (due to the *Peter-Weyl Theorem*), so that the other quantity

$$\sum_{j=1}^{n} \gamma_j \sum_{i=1}^{n} \beta_{ij} \sum_{\alpha_i^j \in A} (f, \chi_{\alpha_i^j}) \chi_{\alpha_i^j}$$

(in the same Definition above) should be seen as the *total contribution of* $\{\chi_{\alpha_i^j}\}_{\alpha_i^j\in A}$ *in* $L^2(G)$ *in its bid to attain* f. Thus the informed reader would see that the inequality in Definition 3.1 above simply gives a measure of how close to the completeness (of $\{\sqrt{d(\lambda)}u_{ij}^{\lambda}\}$) is the orthonormal set $\{\chi_{\alpha_i^j}\}_{\alpha_i^j\in A}$.

The *standard Peter-Weyl orthonormal basis* $\{\sqrt{d(\lambda)}u_{ij}^{\lambda}\}$ used in the above Definition 3.1 may be replaced by any other known complete orthonormal set $\{v_\mu\}_{\mu\in B}$ in $L^2(G)$ while the concept of a semicomplete orthonormal set (for $\{\chi_{\alpha_i^j}\}_{\alpha_i^j\in A}$) could also be defined for an arbitrary Hilbert space, H, so as to have what may be generally called *a semicomplete orthonormal set in* H *with respect to (the complete orthonormal set)* $\{v_\mu\}_{\mu\in B}$ *in* H. If in this general case the set $\{v_\mu\}_{\mu\in B}$ in H is also not necessarily complete, we may arrive at the notion of *a relative semicomplete orthonormal set* for $\{\chi_{\alpha_i^j}\}_{\alpha_i^j\in A}$ in H with respect to $\{v_\mu\}_{\mu\in B}$ in H. Thus Definition 3.1 may therefore be seen as giving *semicompleteness of* $\{\chi_{\alpha_i^j}\}_{\alpha_i^j\in A}$ *in* $L^2(G)$ *with respect to the standard Peter-Weyl orthonormal basis* $\{\sqrt{d(\lambda)}u_{ij}^{\lambda}\}$.

It is clear that every complete orthonormal set in $L^2(G)$ (or in any Hilbert space, H) is automatically semicomplete; simply choose $A = \widehat{G}$, $\gamma_j = \beta_{ij} = 1$, but not conversely. An inductive method of immediately constructing a semicomplete orthonormal set in a compact group is by a method of *selective omission* of some number of members in any known complete (or of the *standard Peter-Weyl*) orthonormal set with a *controlled bound*. The control of the bound in the method of *selective omission* would be achieved using the *Riemann-Lebesgue Lemma*.

This method, as contained in the following, equally gives an *existence* argument for the concept of a semicomplete orthonormal set in a compact group.

3.2 Lemma. (*Existence of a semicomplete orthonormal set: the standard Riemann-Lebesgue orthonormal set on the Torus,* $\mathbb{T}$) *There exist* $\lambda_0 \in \widehat{\mathbb{T}}$ *for which*

$$|\langle f, u_{km}^{\lambda}\rangle| < \frac{\epsilon}{d(\lambda_0)^2},$$

for every $f \in L^2(\mathbf{T})$, $|\lambda| \geq |\lambda_0|$ *and* $1 \leq k, m \leq d(\lambda_0)$. *Moreover,*

$$\{\sqrt{d(\lambda)}u_{ij}^{\lambda} : \lambda \in \widehat{G} \setminus \{\lambda_0\},\ 1 \leq i, j \leq d(\lambda)\}$$

is a semicomplete orthonormal set on $\mathbf{T}$.

Proof. Since the dual group $\widehat{\mathbf{T}}$ is discrete, so that

$$\lim_{|\lambda|\to\infty} \langle f, u_{ij}^{\lambda}\rangle = \lim_{|\lambda|\to\infty} \widehat{f(\lambda)}_{ij} = 0 \text{ (by the } \textit{Riemann-Lebesgue Lemma}),$$

it follows that there are (infinitely) many possible $\lambda \in \widehat{G}$ (choose such one λ_0) with $|\lambda| \geq |\lambda_0|$ for which $|\langle f, u_{km}^{\lambda}\rangle| = |\langle f, u_{km}^{\lambda}\rangle - 0| < \frac{\epsilon}{d(\lambda_0)^2}$, for every $f \in L^2(\mathbf{T})$ and $1 \leq k, m \leq d(\lambda_0)$, as required. Hence,

$$\|\sum_{\lambda\in\widehat{\mathbf{T}}} d(\lambda) \sum_{i,j=1}^{d(\lambda)} \langle f, u_{ij}^{\lambda}\rangle u_{ij}^{\lambda} - \sum_{\lambda\in\widehat{\mathbf{T}}\setminus\{\lambda_0\}} d(\lambda) \sum_{i,j=1}^{d(\lambda)} \langle f, u_{ij}^{\lambda}\rangle u_{ij}^{\lambda}\|_2 = \| d(\lambda_0) \sum_{i,j=1}^{d(\lambda_0)} \langle f, u_{ij}^{\lambda_0}\rangle u_{ij}^{\lambda_0}\|_2$$

$\leq d(\lambda_0) \sum_{i,j=1}^{d(\lambda_0)} |\langle f, u_{ij}^{\lambda_0}\rangle| < \epsilon$, for every $f \in L^2(\mathbf{T})$. □

The technique of Lemma 3.2 may be extended as follows. Generally, choose (as assured by the *Riemann-Lebesgue Lemma*) $\lambda_0^{(1)}, \lambda_0^{(2)}, \cdots \in \widehat{\mathbf{T}}$ for which

$$\sum_{k=1}^{\infty} |\langle f, u_{ij}^{\lambda}\rangle| < \frac{\epsilon}{(\sum_{k=1}^{\infty} d(\lambda_0^{(k)}))^2}$$

where $|\lambda| \geq \max\{|\lambda_0^{(1)}|, |\lambda_0^{(2)}|, \cdots\}$ and $f \in L^2(\mathbf{T})$. Then, with proof essentially the same as in Lemma 3.2, the set

$$\{\sqrt{d(\lambda)}u_{ij}^{\lambda} : \lambda \in \widehat{G} \setminus \{\lambda_0^{(1)}, \lambda_0^{(2)}, \cdots\},\ 1 \leq i, j \leq d(\lambda)\}$$

is a semicomplete orthonormal set on $\mathbf{T}$. We shall henceforth refer to the semicomplete orthonormal set

$$\{\sqrt{d(\lambda)}u_{ij}^{\lambda} : \lambda \in \widehat{\mathbf{T}} \setminus \{\lambda_0^{(1)}, \lambda_0^{(2)}, \cdots\},\ 1 \leq i, j \leq d(\lambda)\}$$

as the *standard Riemann-Lebesgue (semicomplete) orthonormal set* on $\mathbf{T}$ (being in correspondence with the *standard Peter-Weyl (complete) orthonormal set*, $\{\sqrt{d(\lambda)}u_{ij}^{\lambda}\}$.)

Other *non-standard* examples of Definition 3.1 may be deduced from the numerous *special functions* of mathematical physics where their corresponding non-zero scalars γ_j and β_{ij} in Definition 3.1 could be calculated from.

3.3 Remarks. In contrast to the zero-subspace $H_{\mathfrak{P}}(\chi_\alpha)$ of Remarks 2.3 we may, in the context of a semicomplete orthonormal set $\{\chi_\alpha\}_{\alpha\in A}$ in a Hilbert space $(H, \langle\cdot,\cdot\rangle)$, consider the subspace

$$H'_{\mathfrak{P}}(\chi_\alpha) := \{x \in H : \ \langle x, \chi_\alpha\rangle = 0, \ (\text{for every } \alpha \in A) \text{ implies } x = 0\},$$

for some orthonormal set $\{\chi_\alpha\}_{\alpha\in A}$ in H. It is clear (from Lemma 2.2) that $H'_{\mathfrak{P}}(\chi_\alpha) = H$ (hence equal to $H(\chi_\alpha)$ and $H_{\mathfrak{P}}(\chi_\alpha)$) if, and only if, $\{\chi_\alpha\}_{\alpha\in A}$ is complete in H and that, when $\{\chi_\alpha\}_{\alpha\in A}$ is semicomplete in H or in $L^2(G)$, both $H_{\mathfrak{P}}(\chi_\alpha)$ and $H'_{\mathfrak{P}}(\chi_\alpha)$ are non-zero: an example may be seen from using the *standard Riemann-Lebesgue orthonormal set* on $\mathbf{T}$. In general, we have the following.

3.4 Lemma. *Let $(H, \langle\cdot,\cdot\rangle)$ denote any Hilbert space. Then*

$$H(\chi_\alpha) \subseteq H'_{\mathfrak{P}}(\chi_\alpha)$$

for any semicomplete orthonormal set $\{\chi_\alpha\}_{\alpha\in A}$ in H.

Proof. Choose any $x \in H(\chi_\alpha)$, then $x = \sum_{\alpha\in A}\langle x, \chi_\alpha\rangle\chi_\alpha$. Now if $\langle x, \chi_\alpha\rangle = 0$, for every $\alpha \in A$, then

$$x = \sum_{\alpha\in A}\langle x, \chi_\alpha\rangle\chi_\alpha = \sum_{\alpha\in A}(0)\chi_\alpha = 0;$$

showing that $x = 0$ as required. □

We shall refer to $H'_{\mathfrak{P}}(\chi_\alpha)$ as the *prime-Parseval subspace* of H and the choice of this term is further reinforced by the following facts.

3.5 Lemma. (*cf.* Lemma 2.2) *Let $\{\chi_\alpha\}_{\alpha\in A}$ denote a semicomplete orthonormal set in a Hilbert space $(H, \langle\cdot,\cdot\rangle)$ and let $x \in H$. Then $x \in H'_{\mathfrak{P}}(\chi_\alpha)$ whenever $\| x \|^2 = \sum_{\alpha\in A} | \langle x, \chi_\alpha\rangle |^2$.*

Proof. If $\| x \|^2 = \sum_{\alpha\in A} | \langle x, \chi_\alpha\rangle |^2$ and $| \langle x, \chi_\alpha\rangle | = 0$ (for every $\alpha \in A$), then $\| x \|^2 = \sum_{\alpha\in A} | \langle x, \chi_\alpha\rangle |^2 = \sum_{\alpha\in A}(0) = 0$; showing that $x = 0$. Hence $x \in H'_{\mathfrak{P}}(\chi_\alpha)$. □

Lemma 3.5 shows the first partial connection between the satisfaction of *Parseval equality*, on one hand, and membership in the *prime-Parseval*

subspace, on the other. The last Lemma may also be seen as saying that the subset of H given as

$$\{x \in H : \| x \|^2 = \sum_{\alpha \in A} | \langle x, \chi_\alpha \rangle |^2, \text{ for any orthonormal set } \{\chi_\alpha\}_{\alpha \in A}\}$$

is also a subset of $H'_{\mathfrak{P}}(\chi_\alpha)$, with clear equality when $\{\chi_\alpha\}_{\alpha \in A}$ is complete. It will be satisfying to also have the reverse inclusion,

$$H'_{\mathfrak{P}}(\chi_\alpha) \subseteq \{x \in H : \| x \|^2 = \sum_{\alpha \in A} | \langle x, \chi_\alpha \rangle |^2, \text{ for any orthonormal set } \{\chi_\alpha\}_{\alpha \in A}\}$$

due to the importance of the Parseval equality in the fine properties of Fourier transform. We shall deal with this concern in Lemma 3.12.

Even though a semicomplete orthonormal set $\{\chi_\alpha\}_{\alpha \in A}$ in $L^2(G)$ (or in a Hilbert space $(H, \langle , \cdot , \rangle)$) may not be dense, as it is generally expected of a complete orthonormal set, we may still however employ this orthonormal set to construct some dense subspaces of $L^2(G)$ (or of a Hilbert space $(H, \langle , \cdot , \rangle)$) as follows. Indeed, the following results on the *Fourier subspace* for $L^2(G)$ are also valid for an arbitrary Hilbert space, $(H, \langle , \cdot , \rangle)$ and for a *relative semicomplete orthonormal set* in H.

3.6 Theorem. *Let G denote a compact and let $\{\chi_{\alpha_i^j}\}_{\alpha_i^j \in A}$ denote a semicomplete orthonormal set on G. Then $L^2(G)(\chi_{\alpha_i^j})$ is topologically dense in $L^2(G)$.*

Proof. Since every $f \in L^2(G)$ may be expanded as

$$\sum_{\lambda \in \widehat{G}} d(\lambda) \sum_{i,j=1}^{d(\lambda)} \langle f, u_{ij}^\lambda \rangle u_{ij}^\lambda$$

(with convergence in the norm of $L^2(G)$) it follows that for $\epsilon > 0$ we have

$$\| f - \sum_{\lambda \in \widehat{G}} d(\lambda) \sum_{i,j=1}^{d(\lambda)} \langle f, u_{ij}^\lambda \rangle u_{ij}^\lambda \|_2 < \frac{\epsilon}{2}.$$

Now

$$\| f - \sum_{j=1}^{n} \gamma_j \sum_{i=1}^{n} \beta_{ij} \sum_{\alpha_i^j \in A} \langle f, \chi_{\alpha_i^j} \rangle \chi_{\alpha_i^j} \|_2 \leq \| f - \sum_{\lambda \in \widehat{G}} d(\lambda) \sum_{i,j=1}^{d(\lambda)} \langle f, u_{ij}^\lambda \rangle u_{ij}^\lambda \|_2$$

$$+ \| \sum_{\lambda\in\widehat{G}} d(\lambda) \sum_{i,j=1}^{d(\lambda)} \langle f, u_{ij}^{\lambda}\rangle u_{ij}^{\lambda} - \sum_{j=1}^{n} \gamma_j \sum_{i=1}^{n} \beta_{ij} \sum_{\alpha_i^j\in A} \langle f, \chi_{\alpha_i^j}\rangle \chi_{\alpha_i^j} \|_2 < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \square$$

In more specific terms we have the following.

3.7 Corollary. *Let G denote a compact group and let $\{\chi_{\alpha_i^j}\}_{\alpha_i^j\in A}$ denote a semicomplete orthonormal set on G. Then every $f \in L^2(G)$ can be expanded as*

$$f = \sum_{j=1}^{n} \gamma_j \sum_{i=1}^{n} \beta_{ij} \sum_{\alpha_i^j\in A} \langle f, \chi_{\alpha_i^j}\rangle \chi_{\alpha_i^j}$$

for some $\gamma_j, \beta_{ij} \in \mathbf{C}$ with convergence in the norm on $L^2(G)$. $\square$

We may refer to the expansion of f in Corollary 3.7 as a *semi-Fourier series expansion* for $f \in L^2(G)$ or H with respect to $\{\chi_{\alpha_i^j}\}_{\alpha_i^j\in A}$. A stronger form of Theorem 3.6 carved in the form of the equivalence of Lemma 2.2 and which generalizes the fact that a mutually orthonormal family $\{\chi_\alpha\}_{\alpha\in A}$ is complete (in a Hilbert space $(H, \langle\cdot,\cdot\rangle)$) if, and only if, $H(\chi_\alpha) = H$ (*cf.* Lemma 2.2) is also possible when the mutually orthonormal family $\{\chi_\alpha\}_{\alpha\in A}$ is semicomplete in H. We prove this below in the special case of $H = L^2(G)$.

3.8 Theorem. *Let G denote a compact group and let $\{\chi_{\alpha_i^j}\}_{\alpha_i^j\in A}$ denote a mutually orthonormal set on G whose Fourier subspace is denoted as $L^2(G)(\chi_{\alpha_i^j})$. Then $L^2(G)(\chi_{\alpha_i^j})$ is topologically dense in $L^2(G)$ if, and only if, $\{\chi_{\alpha_i^j}\}_{\alpha_i^j\in A}$ is semicomplete.*

Proof. That $L^2(G)(\chi_{\alpha_i^j})$ is topologically dense in $L^2(G)$ if $\{\chi_{\alpha_i^j}\}_{\alpha_i^j\in A}$ is semicomplete is the content of Theorem 3.6. Now choose $f \in L^2(G)$, then

$$\| \sum_{\lambda\in\widehat{G}} d(\lambda) \sum_{i,j=1}^{d(\lambda)} \langle f, u_{ij}^{\lambda}\rangle u_{ij}^{\lambda} - \sum_{j=1}^{n} \gamma_j \sum_{i=1}^{n} \beta_{ij} \sum_{\alpha_i^j\in A} \langle f, \chi_{\alpha_i^j}\rangle \chi_{\alpha_i^j} \|_2$$

$$\leq \| \sum_{\lambda\in\widehat{G}} d(\lambda) \sum_{i,j=1}^{d(\lambda)} \langle f, u_{ij}^{\lambda}\rangle u_{ij}^{\lambda} - f \|_2 + \| f - \sum_{j=1}^{n} \gamma_j \sum_{i=1}^{n} \beta_{ij} \sum_{\alpha_i^j\in A} \langle f, \chi_{\alpha_i^j}\rangle \chi_{\alpha_i^j} \|_2$$

$\frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$(using the *Peter-Weyl theorem* and Corollary 3.7, respectively). $\square$

This Theorem would enable us to see the *Peter-Weyl series expansion* of every $f \in L^2(G)$, given as

$$f = \sum_{\lambda\in\widehat{G}} d(\lambda) \sum_{i,j=1}^{d(\lambda)} \langle f, u_{ij}^{\lambda}\rangle u_{ij}^{\lambda}$$

(with convergence in the L^2–norm), as the restriction of the *semi-Fourier series expansion*

$$f = \sum_{j=1}^{n} \gamma_j \sum_{i=1}^{n} \beta_{ij} \sum_{\alpha_i^j \in A} \langle f, \chi_{\alpha_i^j} \rangle \chi_{\alpha_i^j}$$

to the *standard Peter-Weyl (complete) mutually orthonormal set* $\{\sqrt{d(\lambda)}u_{ij}^{\lambda}\}$. Indeed Theorem 3.8 leads to the same conclusion for the *prime-Parseval subspace* $L^2(G)'_{\mathfrak{P}}(\chi_{\alpha_i^j})$.

3.9 Corollary. *Let G denote a compact group and let $\{\chi_{\alpha_i^j}\}_{\alpha_i^j \in A}$ denote a mutually orthonormal set on G. Then $L^2(G)'_{\mathfrak{P}}(\chi_{\alpha_i^j})$ is topologically dense in $L^2(G)$ if, and only if, $\{\chi_{\alpha_i^j}\}_{\alpha_i^j \in A}$ is semicomplete.*

Proof. Consider Lemma 3.4 in the light of Theorem 3.8. □

The inclusion $L^2(G)(\chi_{\alpha_i^j}) \subseteq L^2(G)'_{\mathfrak{P}}(\chi_{\alpha_i^j})$ of Lemma 3.4, when combined with both Theorem 3.7 and Corollary 3.9, implies the following.

3.10 Corollary. *$L^2(G)(\chi_{\alpha_i^j})$ is topologically dense in $L^2(G)'_{\mathfrak{P}}(\chi_{\alpha_i^j})$.* □

The converse of Lemma 3.5 is now immediate for both $L^2(G)'_{\mathfrak{P}}(\chi_{\alpha_i^j})$ and (even) $H'_{\mathfrak{P}}(\chi_{\alpha_i^j})$ in any arbitrary Hilbert space, $(H, \langle\cdot,\cdot\rangle)$.

3.11 Lemma.(*cf.* Lemma 2.2) *Let G denote a compact group and let $\{\chi_\alpha\}_{\alpha\in A}$ denote a mutually orthonormal set on G. Then $f \in L^2(G)'_{\mathfrak{P}}(\chi_\alpha)$ if, and only if, $\| f \|_2^2 = \sum_{\alpha\in A} | \langle f, \chi_\alpha \rangle |^2$.*

Proof. Let $f \in L^2(G)'_{\mathfrak{P}}(\chi_\alpha)$. We may take $f \in L^2(G)(\chi_\alpha)$ due to Corollary 3.10; so that $f = \sum_{\alpha\in A}\langle f, \chi_\alpha\rangle\chi_\alpha$. Hence

$$0 = \| f - \sum_{\alpha\in A} \langle f, \chi_\alpha\rangle\chi_\alpha \|_2^2 = \| f \|_2^2 - \sum_{\alpha\in A} | \langle f, \chi_\alpha\rangle |^2,$$

as required. □

Hence, the *prime-Parseval subspace* $L^2(G)'_{\mathfrak{P}}(\chi_{\alpha_i^j})$ may finally be seen (for some orthonormal set $\{\chi_{\alpha_i^j}\}_{\alpha_i^j\in A}$) as

$$L^2(G)'_{\mathfrak{P}}(\chi_{\alpha_i^j}) = \{f \in L^2(G) : \| f \|_2^2 = \sum_{\alpha_i^j \in A} | \langle f, \chi_{\alpha_i^j}\rangle |^2\}$$

We now have enough preparation to introduce a Fourier transform $f \mapsto \widehat{f}$ on the *prime-Parseval subspace*, $L^2(G)'_{\mathfrak{P}}(\chi_{\alpha_i^j})$.

Consider $f \in L^2(G)$ and for every $\alpha \in A$ define the matrix $\widehat{f}(\alpha)$ whose entries are given as

$$\widehat{f}(\alpha)_{ij} := \widehat{f}(\alpha_i^j).$$

That is, $\widehat{f}(\alpha)_{ij} := \langle f, \chi_{\alpha_i^j}\rangle$, for $1 \leq i,j \leq n$. The Parseval inequality of $L^2(G)'_{\mathfrak{P}}(\chi_{\alpha_i^j})$ (in Lemma 3.11) therefore becomes $\| f \|_2^2 = \sum_{\alpha\in A} \| \widehat{f}(\alpha) \|^2$, for every $f \in L^2(G)'_{\mathfrak{P}}(\chi_{\alpha_i^j})$, where $\| \widehat{f}(\alpha) \|^2$ is the Hilbert-Schimdt norm of the matrix

$$\widehat{f}(\alpha) = (\widehat{f}(\alpha)_{ij})_{i,j=1}^n = (\widehat{f}(\alpha_i^j))_{i,j=1}^n.$$

In other words, and in terms of our choice of indexing A, we have

$$\| f \|_2^2 = \sum_{i,j=1}^n \sum_{\alpha_i^j\in A} \| \widehat{f}(\alpha_i^j) \|^2,$$

for $f \in L^2(G)'_{\mathfrak{P}}(\chi_{\alpha_i^j})$.

3.12 Definition. Set $L^2(A)$ as the space of matrix-valued functions φ on A with values in $\bigcup_{n=1}^{\infty} M_n(\mathbf{C})$ satisfying
(*i*) $\varphi(\alpha_i^j) \in M_n(\mathbf{C})$ for all $\alpha_i^j \in A$ and
(*ii*) $\sum_{i,j=1}^n \sum_{\alpha_i^j\in A} \| \varphi(\alpha_i^j) \|^2 < \infty$. □

The inner product $(\cdot,\cdot)$ on $L^2(A)$ given as

$$(\varphi,\psi) := \sum_{i,j=1}^n \sum_{\alpha_i^j\in A} tr(\varphi(\alpha_i^j)\psi(\alpha_i^j)^*),$$

$\varphi,\psi \in L^2(A)$ converts $(L^2(A),(\cdot,\cdot))$ into a Hilbert space. We can then establish a connection between the *prime-Parseval subspace* $L^2(G)'_{\mathfrak{P}}(\chi_{\alpha_i^j})$ (which is a Hilbert subspace of $L^2(G)$) and $L^2(A)$.

3.13 Theorem. (Fourier image of the *prime-Parseval subspace*) *Let G denote a compact group and let $\{\chi_{\alpha_i^j}\}_{\alpha_i^j\in A}$ denote a semicomplete mutually orthonormal set on G. Then the map*

$$\mathcal{H} : L^2(G)'_{\mathfrak{P}}(\chi_{\alpha_i^j}) \to L^2(A) : f \mapsto \mathcal{H}(f) := \widehat{f}$$

is an isometry of $L^2(G)'_{\mathfrak{P}}(\chi_{\alpha_i^j})$ onto $L^2(A)$. □

Theorem 3.13 is very familiar when the semicomplete mutually orthonormal set $\{\chi_{\alpha_i^j}\}_{\alpha_i^j\in A}$ is the complete mutually orthonormal set $\{\sqrt{d(\lambda)}u_{ij}^{\lambda}\}$. We do not yet know the general connection between the set A and the dual group $\widehat{G}$, except in the special cases of the *standard Riemann-Lebesgue (semicomplete) orthonormal sets* on $\mathbf{T}$. We however see A as a general form of $\widehat{G}$ which

may take the usual form of $\widehat{G}$ in specific cases. If we set

$$H_i^\alpha := \sum_{j=1}^{n} \mathbf{C}\chi_{\alpha_i^j},$$

for $\alpha = \alpha_i^j \in A$ and $i \in \{1, \cdots, n\}$, then the Hilbert subspace $L^2(G)'_{\mathfrak{P}}(\chi_{\alpha_i^j})$ of $L^2(G)$ has the direct-sum decomposition

$$L^2(G)'_{\mathfrak{P}}(\chi_\alpha) = \bigoplus_{\alpha \in A} \bigoplus_{i=1}^{n} H_i^\alpha.$$

The results of this section laid a foundation for harmonic analysis of the *prime-Parseval subspace* $H'_{\mathfrak{P}}(\chi_{\alpha_i^j})$ with respect to a semicomplete orthonormal set $\{\chi_{\alpha_i^j}\}_{\alpha_i^j \in A}$ in a Hilbert space, H. Having considered the case of the Hilbert space $L^2(G)$, for a compact group G, in this section it will a delight to use these foundational results (on both $H'_{\mathfrak{P}}(\chi_{\alpha_i^j})$ and $L^2(G)'_{\mathfrak{P}}(\chi_{\alpha_i^j})$) in the understanding of further properties of $L^2(G)'_{\mathfrak{P}}(\chi_{\alpha_i^j})$ in the full sight of the semicompleteness of $\{\chi_{\alpha_i^j}\}_{\alpha_i^j \in A}$. We shall give a very short introduction to this type of study for a connected semisimple Lie group in the next section.

It is clear from Lemma 3.2, for *standard (Riemann-Lebesgue)* examples of a semicomplete orthonormal set in an arbitrary Hilbert space $(H, \langle\cdot,\cdot\rangle)$ or in $L^2(G)$, that the non-zero constants γ_j and β_{ij} would always be $\gamma_j = \beta_{ij} = 1$ for $1 \leq i, j \leq \mid \widehat{G} \setminus \{\lambda_0^{(1)}, \lambda_0^{(2)}, \cdots\} \mid$. However, for *non-standard* examples of a semicomplete orthonormal set in an arbitrary Hilbert space $(H, \langle\cdot,\cdot\rangle)$ or in $L^2(G)$, the *semi-Fourier series expansion* of Corollary 3, 7 may have to be broken down in order for general expressions for γ_j and β_{ij} to be known. A first result along this line is the following.

3.14 Lemma. *Let $\{\chi_{\alpha_i^j}\}_{\alpha_i^j \in A}$ denote a semicomplete orthonormal set in a Hilbert space $(H, \langle\cdot,\cdot\rangle)$ and let $x \in H$. Then*

$$\langle x, \chi_{\alpha_i^j}\rangle = \gamma_i\beta_{ii}\langle x, \chi_{\alpha_i^j}\rangle,$$

for $1 \leq i \leq n$. In particular, $\gamma_i\beta_{ii} = 1$.

Proof. We have that $\langle x, \chi_{\alpha_k^j}\rangle = \sum_{j=1}^{n} \gamma_j \sum_{i=1}^{n} \beta_{ij} \sum_{\alpha_i^j \in A} \langle x, \chi_{\alpha_i^j}\rangle\langle \chi_{\alpha_i^j}, \chi_{\alpha_k^j}\rangle$. Due to the orthogonality of the set $\{\chi_{\alpha_i^j}\}_{\alpha_i^j \in A}$ the above equality reduces to $\langle x, \chi_{\alpha_i^j}\rangle = \gamma_i\beta_{ii}\langle x, \chi_{\alpha_i^j}\rangle$, for $1 \leq i \leq n$ as required.

Now $(1 - \gamma_i\beta_{ii})\langle x, \chi_{\alpha_i^j}\rangle = 0$ from where we have $\gamma_i\beta_{ii} = 1$. □

§4. K-semicomplete orthonormal set in a semisimple Lie group.

The success in §3. of the use of the notion of a *semicomplete orthonormal* set in the harmonic analysis of a compact group, culminating in the extraction and elucidation of the *prime-Parseval subspace* as well as its Fourier image, shows the central importance and the correct us of *Parseval equality* and the concept of *completeness* (of an orthonormal set) in the abstract Peter-Weyl theory of a compact group and in the understanding of the hitherto unknown subspaces of $L^2(G)$ under the influence of the Fourier transform. This study (which led us to the consideration of the *prime-Parseval subspace* $L^2(G)'_{\mathfrak{P}}(\chi_{\alpha_i^j})$ corresponding to a semicomplete orthonormal set $\{\chi_{\alpha_i^j}\}_{\alpha_i^j \in A}$ on G) is reminiscence of and may be compared with the extraction and harmonic analysis of the *Schwartz algebra* in the L^2–theory of semisimple Lie groups which was started in the Yale thesis [1(a.)] of James Arthur (continued and completed in two later manuscripts, [1(b.)] and [1(c.)]). In a more recent publication, harmonic analysis of other spaces of functions on semisimple Lie groups, namely of the space of *spherical convolutions*, has been introduced in [3.] leading to the explicit construction of the corresponding *Plancherel formula* for such functions. The present paper has also introduced the *Fourier* and *prime-Parseval subspaces* of $L^2(G)$ (or of any arbitrary Hilbert space, $(H, \langle\cdot,\cdot\rangle)$).

Having shown in §3. the essential importance of the Parseval equality (which is the precursor of the Plancherel formula) in the consideration of the actual subspace of $L^2(G)$ under the natural action of the Fourier transform, we shall here consider studying the same theory (of a semicomplete orthonormal set) but for all semisimple Lie groups, having removed the impediments posed by the *completeness* for orthonormal sets on such Lie groups.

It is well-known that orthonormal sets (of functions and polynomials) are numerous and readily available in the L^2–space (and more recently in some distinguished subspaces of the L^{2n}–spaces [4.]) of semisimple Lie groups. Indeed every semisimple Lie group has its corresponding orthonormal set, an example is $G = SL(2, \mathbf{R})$ and its *Legendre functions*.

Even though these sets of orthonormal functions and polynomials are central to harmonic analysis on these groups, their direct importance in or contribution to the decomposition of (sub-)spaces of $L^2(G)$ or expansion of their members is not yet known. In the outlook of the present section (and

of the entire paper) any orthonormal set on a semisimple Lie group known to have been $K-$*semicomplete* (in the sense to be soon made precise) could be a basis of some subspaces of $L^2(G)$.

4.1 Definition. ($K-$semicomplete orthonormal set) *Let $G = KAN$ denote the Iwasawa decomposition of a connected semisimple Lie group G with finite center. An orthonormal set $\{\chi_\alpha\}_{\alpha\in A}$ on G is said to be $K-$semicomplete whenever its restriction to K, written as $\{(\chi_\alpha)_{|K}\}_{\alpha\in A}$, is a semicomplete orthonormal set in $L^2(K)$.* □

It is relatively easy to construct a $K-$*semicomplete orthonormal set* on any connected semisimple Lie group G, from any given semicomplete orthonormal set on K as follows.

4.2 An example. Choose any of the numerous orthonormal sets $\{\xi_\alpha\}_{\alpha\in A}$ in $L^2(K)$ as constructed in §3. and, for every $x = kan \in G$, define the map $\chi_\alpha : G \to \mathbf{C}$ as

$$\chi_\alpha(x) = \chi_\alpha(kan) := e^{f(an)}\xi_\alpha(k),$$

where $f : AN \to \mathbf{C}$ satisfies
(i) $f(1) = 0$,
(ii) $\int_{AN} e^{2\Re(f(an))}dadn = 1$ and
(iii) $\int_{AN} g(kan)(e^{\overline{f(an)}+f(a_1n_1)})dadn = g(k)$, for $g \in L^2(G)$, $a_1 \in A$, $n_1 \in N$ and the normalized Haar measures da and dn on A and N, respectively.

Proof. Observe that since

$$\chi_\alpha(x) = \chi_\alpha(kan) := e^{f(an)}\xi_\alpha(k),$$

then for any $k \in K$

$$\chi_\alpha(k) = \chi_\alpha(k\cdot 1\cdot 1) := e^{f(1\cdot 1)}\xi_\alpha(k) = \xi_\alpha(k).$$

For any $\alpha_1, \alpha_2 \in A$, we have

$$\langle \chi_{\alpha_1}, \chi_{\alpha_2}\rangle = \int_K \Big(\int_{AN} e^{2\Re(f(an))}dadn\Big)\xi_{\alpha_1}(k)\overline{\xi_{\alpha_2}(k)}dk = \langle \xi_{\alpha_1}, \xi_{\alpha_2}\rangle$$

and

$$\| \chi_\alpha \|_2^2 = \int_K \Big(\int_{AN} e^{2\Re(f(an))}dadn\Big) \mid \xi_{\alpha_1}(k) \mid^2 dk = \| \xi_\alpha \|_2^2 = 1;$$

showing that $\{\chi_\alpha\}_{\alpha\in A}$ is an orthonormal set on G. Its $K-$semicompleteness is also shown as follows. For a pre-assigned $\epsilon > 0$, we have that

$$\| \sum_{\lambda\in\widehat{G}} d(\lambda) \sum_{i,j=1}^{d(\lambda)} (g, u_{ij}^\lambda)u_{ij}^\lambda - \sum_{j=1}^{n}\gamma_j \sum_{i=1}^{n}\beta_{ij} \sum_{\alpha_i^j\in A} (g, \chi_{\alpha_i^j})\chi_{\alpha_i^j} \|_2$$

$$=\| \sum_{\lambda\in\widehat{G}} d(\lambda) \sum_{i,j=1}^{d(\lambda)} (g, u_{ij}^{\lambda}) u_{ij}^{\lambda} - \int_K [\int_{AN} g(kan)(e^{\overline{f(an)}+f(a_1n_1)}) da dn] \cdot$$

$$\sum_{j=1}^{n} \gamma_j \sum_{i=1}^{n} \beta_{ij} \sum_{\alpha_i^j \in A} \overline{\xi_{\alpha_i^j}(k)} dk \, \xi_{\alpha_i^j} \|_2$$

$$=\| \sum_{\lambda\in\widehat{G}} d(\lambda) \sum_{i,j=1}^{d(\lambda)} (g, u_{ij}^{\lambda}) u_{ij}^{\lambda} - \sum_{j=1}^{n} \gamma_j \sum_{i=1}^{n} \beta_{ij} \sum_{\alpha_i^j \in A} (g, \xi_{\alpha_i^j}) \xi_{\alpha_i^j} \|_2 < \epsilon. \ \square$$

For any K-semicomplete orthonormal set $\{\chi_\alpha\}_{\alpha\in A}$ on G the corresponding *Fourier subspace* $L^2(G)(\chi_\alpha)$ of $L^2(G)$ is also given as

$$L^2(G)(\chi_\alpha) := \{f \in L^2(G) : \ f = \sum_{\alpha\in A} (f, \chi_\alpha)\chi_\alpha\}$$

while the *prime-Parseval subspace* is

$$L^2(G)'_{\mathfrak{P}}(\chi_\alpha) := \{f \in L^2(G) : \ (f, \chi_\alpha) = 0 \text{ (for every } \alpha \in A) \text{ implies } f = 0\}.$$

Clearly $L^2(K)'_{\mathfrak{P}}(\sqrt{d(\lambda)}u_{ij}^{\lambda}) = L^2(K)$ (from Lemma 2.2 (*ii*)), both subspaces $L^2(K)(\chi_\alpha)$ and $L^2(K)'_{\mathfrak{P}}(\chi_\alpha)$ are topologically dense in $L^2(K)$ (from Theorems 3.6 and 3.8 and Corollary 3.9) and there exists an isometry of $L^2(K)'_{\mathfrak{P}}(\chi_\alpha)$ onto $L^2(A)$ (from Theorem 3.13). We shall resume the study of the subspaces $L^2(G)(\chi_\alpha)$ and $L^2(G)'_{\mathfrak{P}}(\chi_\alpha)$ (for connected semisimple Lie groups, G) in another paper.

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